DISTRIBUTION OF CERTAIN STATISTICS EXPRESSIBLE AS PRODUCT OF INDEPENDENT CHI-SQUARES—A REVIEW

By

O.P. BAGAI AND SADHANA TIWARI

Punjab University, Chandigarh

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INTRODUCTION

Consider a p-variate normal population with dispersion matrix Σ. Wilks [28] defined the det. $|\Sigma|$ to be a measure of the spread of the observations and named it to be the generalized variance.

Let det. $|S|$ be the sample generalized variance with n d. f. Further, let $A=nS$ and let k_i^2 $(i=1, 2,...p)$ be the roots of the determinantal equation :

$$|T-k^2\Sigma|=0 \qquad ...(1.1)$$

where T is the non-centrality matrix of S,

When $k_i^2=0$ $(i=2,3,....p)$ and $k_1^2\neq 0$, the h^{th} order moment of $|A|$ in the non-central linear case is given by Anderson [1] as follows :

$$E(|A|^h)=|\Sigma|^h\; 2^{ph} \exp\left(-k_1^2/2\right) \prod_{i=2}^{p} \frac{\Gamma\left(\frac{n-i+1}{2}+h\right)}{\Gamma\left(\frac{n-i+1}{2}\right)} \sum_{\beta=0}^{\infty}$$

$$\left[\frac{\left(k_1^2\right)^{\beta}\,\Gamma\left(\frac{n}{2}+h+\beta\right)}{2^{\beta}\;\beta\,!\,\Gamma\left(\frac{n}{2}+\beta\right)}\right] \qquad ...(1.2)$$

with the help of which Bagai [2] deduces that

$$|A|=|\Sigma|\prod_{i=1}^{p} v_i \qquad ...(13).$$

where ν_i' s are all independent and ν_1 has the non-central chi-square distribution with n degrees of freedom (d. f) and non-centrality parmeter k_1^2 and ν_i (i=2, 3,..., p) has the central chi-square distribution with (n−i+1) d. f.

Consider next the central distribution of the ratio, ϕ, of $|S|$ to any of its principal minors of order t, *i.e.*,

$$\phi=\frac{|s_{ij}|}{|s_{qr}|} \qquad \left(\begin{array}{l} i, j=1, 2,\ldots,p \\ q, r=1, 2,\ldots, t :< p \end{array}\right) \quad \ldots(1.4)$$

The h^{th} order moment of ϕ is given by Wilks [28] in modified form as follows:

$$E(\phi^h)=B^{-h} \prod_{j=1}^{p-t} \left[\frac{\Gamma\left(\frac{n-j-t+1}{2}+h\right)}{\Gamma\left(\frac{n-j-t+1}{2}\right)}\right] \quad \ldots(1.5)$$

where B is an expression involving population constants.

Thus, as in the above cases, we obtain with little algebra the following :

$$2^{p-t} B\phi=\prod_{j=1}^{p-t} \nu_j \quad \ldots(1.6)$$

where ν_j (j=1, 2,..., $p-t$) are all independent and ν_j is distributed as a central chi-square with $(n-j-t+1)$ d.f.

Further in multivariate analysis of variance (MANOVA) Pillai [25] concludes that the three tests of hypotheses : (*i*) equality of two dispersion matrices, (*ii*) equality of the p-dimensional mean vectors, and (*iii*) the independence between a p—set and a q—set of variates, depend, when the respective hypotheses to be tested are true, only on the roots θ_i, and ϕ_i (i=1,2,...,l), respectively, of the determinantal equations :—

$$|A-\theta(A+C)|=0 \quad \ldots(1.7)$$

and

$$|A-\phi C|=0 \quad \ldots(1.8)$$

where A and C are independent sum of product (S.P) matrices based on random samples of observations with n_1 and n_2 d.f. respectively and are defined differently for different hypotheses. For instance, (*i*) in testing the hypothesis of "equality of two dispersion matrices

Σ_1 and Σ_2" n_1 and n_2 are the d.f. of the two sample dispersion matrices A and C, respectively; (*ii*) in testing the hypothesis of "equality of p-dimensional mean vectors, n_1 and n_2 are the d.f. of the between and the within S.P. matrices A and C, respectively, and (*iii*) in testing the hypothesis of the independence between a $p-$set and a $q-$set of variates" $(p \leqslant q)$, $n_1 = q$ and $n_2 = N-1$ are the d.f. of the independent S.P. matrices $A(\equiv W_{qq}\, W_{pq}^{-1}\, W_{qp})$ and $C(\equiv W_{pp})$, respectively, where (*a*) N is the size of the samples of observations drawn from a $(p+q)$ variate normal population with covariance matrix Σ, and (*b*) W_{pp} is the S.P matrix of the sample observations of the p-set of variates, W_{qq} that on the q-set and W_{pq} that between the observations on the p-set and those on the q-set,

For sufficiently large n, (Nanda [24], Bagai [3]), the common standard form of the joint limiting distribution of eigenroots

$$\theta_i \left(= \frac{c_i}{n}\right) \text{or } \phi_i \left(= \frac{c_i}{n}\right)$$

of the determinantal equations (1.7) and (1.8), under the respective null hypothesis, is as follows :

$$K(l, m) \prod_{i=1}^{l} c_i^m \exp\left(-\sum_{i=1}^{l} c_i\right) \prod_{i=2}^{l} \prod_{j=1}^{i-1} (c_i - c_j) \prod_{i=1}^{l} dc_i \quad \ldots(1.9)$$

where $0 \leqslant c_1 \leqslant c_2 \leqslant \ldots \leqslant c_l < \infty$ and the values of m, n for the respective null hypotheses are as follows :

(*i*) $l = p,\quad m = \dfrac{n_1 - p - 1}{2},\quad n = \dfrac{n_2 - p - 1}{2}$

(*ii*) If $p \leqslant n_1$, $l = p$, then

$$m = \frac{n_1 - p - 1}{2}, \quad n = \frac{n_2 - p - 1}{2}$$

If $p > n_1$, $l = n_1$, then

$$m = \frac{p - n_1 - 1}{2}, \quad n = \frac{n_2 - p - 1}{2}$$

(*iii*) Same as (*ii*)

and $$K(l, m) = \pi^{l/2} / \left[\prod_{i=1}^{l} \Gamma\left(\frac{2m+i+1}{2}\right) \Gamma\left(\frac{i}{2}\right)\right]$$

Both the statistics U_l and Y_l (Bagai [3]) defined as follows :

$$U_l=\frac{|A|}{|A+C|}=\prod_{i=1}^{l}\theta_i=\prod_{i=1}^{l}\phi_i(1+\phi_i)^{-1} \quad ...(1.10)$$

and

$$Y_l=\frac{|A|}{|C|}=\prod_{i=1}^{l}\theta_i(1-\theta_i)^{-1}=\prod_{i=1}^{l}\phi_i \quad ...(1.11)$$

under the hypotheses (*i*), (*ii*) and (*iii*) take the following form in the limiting case :

$$Y_l=U_l=\prod_{i=1}^{l}(2c_i)=\prod_{i=1}^{l}\nu_i \quad ...(1.12)$$

where $\nu_i(i=1, 2, .., l)$ are all independent and ν_i is distributed as a central chi-square variable with $(2m+i+1)$ d.f.

To make the treatment, therefore, more general, we take up, in the sequal, the statistic

$$V_p=\prod_{j=0}^{p-1}\nu_j \quad ...(1.13)$$

where the variate ν_0 is non-central chi-square with n d.f. and non-centrality parameter k^2 and the variate ν_j $(j=1, 2, \quad, p-1)$ is central chi-square with $(n-j)$ d.f.

To find the distribution of statistic V_p there have been to this date three main approaches :

The first approach has been due to Wilks [28] via the method of direct multiple integrals. The same approach was later adopted by Bagai [2], [3], [4], [5], [9]. He obtained the exact distribution of V_p, both in the central as well as non-central linear cases for $p=2(1)$ 12. His method led him to arrive at multiple sum representations for higher values of p.

The second approach, initiated by Nair [23], is through the method of inverse Mellin transforms. Using this method, Consul [11] evaluated the exact densities for $p=2$, 3 and 4 in the form of standard functions and for $p=5$, 6 and 7 in the form of integral representations. His results agree with those of Bagai [2] and Wilks [28].

Mathai and Saxena [22], Mathai and Rathie [21], Mathai [19] and Bagai [6], [7], [8] again use the inverse Mellin transform technique, but obtain the distribution of the same statistic, V_p, in terms of Meijer G—function. Bagai [6] [8] obtains also reduction formulae which help in connecting the G—function of some order with that of order smaller by unity. This proves handy to him in arriving at particular cases. Their results in particular cases are also in the form of hypergeometric functions or same other numerically tractable standard functions.

The third approach is by Herz [15] through the method of Zonal polynomials. James [18] and Constantine [10] dealt with the distribution of V_p in the non-central case via the method of Zonal polynomials.

We discuss very briefly the first two approaches in Section 3. Only salient steps have been indicated and much has been left for the reader to go through the literature referred to in the bibliography provided in the end. The third approach is not being discussed here as it involves a good amount of mathematical abstraction and sophistication. For this approach, the reader is referred to Mathai [20] who provides a detailed discussion and it will be of no use to re-record here. Further, he has commented that the method of evaluating the distributions with the help of Zonal polynomials is not a powerful method because Zonal polynomials of all orders are not available. Hence the exact density and the corresponding distribution function cannot be fully utilized to compute the exact percentage points. Besides this, the method does not yield mathematically elegent representations which could be convenient for computational purposes.

We first list in Section 2 some important preliminary results and definite integrals which have been used in Section 3.

2. Some Preliminary Results

Some important results and definite integrals, either worked out by researchers themselves or borrowed from standard books, are as under :

(*i*) Legendre's duplication formula (Whittaker and Watson [27]) for the gamma functions

$$\Gamma(n)\ \Gamma(n+\tfrac{1}{2})=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad ...(2.1)$$

(*ii*) Consider,

$$L_r(a)=2\int_0^\infty x^{2r+1}\exp\left(-x^2-\frac{a}{x}\right)dx\ ;\ a>0 \qquad (2.2)$$

Originally, Bagai [2] [3] worked out with the value of L_0 (a) by finding the differential equation satisfied by L_0 (a) and solved it by the standard Frobeneous series method. Later, Consul generalized it by differentiating and integrating L_0 (a) successfully with respect to 'a' and concluded that

$$L_r(a)=-2(\gamma+\log a)\frac{a^{2r+2}}{(2r+2)!}\,{}_0F_2\left[\ ;r+\frac{3}{2},\ r+2\ ;-\frac{a^2}{4}\right]$$

$$+\sum_{i=1}^{r}\frac{(r-i)!\,a^{2i}}{2i!}+2\sum_{i=r+1}^{\infty}$$

$$\left\{\frac{(-1)^{i-r-1}}{(i-r-1);}\frac{a^{2i}}{2i!}\left[\psi(2i)-\frac{1}{2}\psi(i-r-1)\right]\right\}$$

$$-a\Gamma\left(r+\frac{1}{2}\right){}_0F_2\left[\ ;-r+\frac{1}{2},\frac{3}{2}\ ;-\frac{a^2}{4}\right] \qquad ...(2.2\,a)$$

where ${}_pF_t$ is the generalized hypergeometric function and $\sum_{i=0}^{r}(r-i)\,!$ is to be treated as unity when $r=0$, γ is the Euler's constant and ψ (r) is defined as follows :

$$\psi(r)=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{r},\quad r\neq 0$$
$$=0\qquad\qquad ,\quad r=0 \qquad ...(2.2b)$$

(*iii*) For any n, Erdélyi ([14] p. 283) gives

$$\int_0^\infty t^{n-1}e^{-(pt+a/t)}\,dt=2(a/p)^{n/2}K_n\,(2\sqrt{ap}\,);\ \text{Re}\ (a)>0 \qquad ...(2.3)$$

where $K_n(\cdot)$ is the modified Bessel function of general order n. Erdélyi ([13] pp 5, 9, 10]) gives the values of $K_n(.)$ for various values of n. Reader is referred to look for himself the values of K_n (.) for n being non-integer, integer, half times an odd integer and $\frac{1}{2}$. All these values of $K_n(.)$ have been employed by researchers in obtaining the distribution of the statistic, V_p, for particular values of p.

(*iv*) Consider next the Mellin transform pair (Titchmarsh, [26], p. 7) $f(x)$ and $F(s)$ related as follows :

$$F(s)=\int_0^\infty f(x)\, x^{s-1}\, dx$$

where s is a complex variable, and

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s}\, ds \qquad \ldots(2.4)$$

Further, if $f_1(x)$ and $f_2(x)$ are inverse Mellin transforms of $F_1(s)$ and $F_2(s)$, respectively, the inverse Mellin transform of the product $F_1(s).\, F_2(s)$ is given (Titchmarsh [26] p. 52) as below :

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(s)\, F_2(s)\, x^{-s} ds=\int_0^\infty f_1(u)\, f_2(x/u)\frac{du}{u} \qquad \ldots(2.4\ a)$$

which can be further generalized to the product of more than two functions.

In particular, Watson's Mellin transform (Titchmarsh [26] p. 197) is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, 2^s\Gamma(s/2)\Gamma\left(\frac{s}{2}+n\right) ds=2^{-n+2}\, x^n K_n(x) \qquad .. (2.4\ b)$$

where K_n (.) is the modified Bessel function of order n.

The form (2.4*b*) is equivalent to

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} u^{-t}\, 2^{2t}\, \Gamma(t+\alpha)\Gamma(t+\alpha+n)\, dt=2^{-(n+2\alpha-1)}u^{\frac{n}{2}+\alpha}\, K_n\sqrt{u}) \qquad \ldots(2.4\ c)$$

for $s=2(t+\alpha)$ and $x=\sqrt{u}$.

(*v*) Erdélyi ([13], p. 207) defines the Meijer G-function as follows :

$$G^{m,\,n}_{p,\,q}\left(x \,\middle|\, \begin{matrix} a_1,\, a_2,\ldots,\, a_p \\ b_1,\, b_2,\ldots,\, b_q \end{matrix}\right)=\frac{1}{2\pi i}\int_L \frac{x^{-s}\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}\, ds \qquad \ldots(2.5)$$

where an empty product is interpreted as one ; $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; and the parameters are such that no pole of $\Gamma(b_j+s)$, $j=1, 2, \ldots, m$ coincides with any pole of $\Gamma(1-a_k-s)$, $k=1, 2 \ldots, m$. L is a suitable contour enclosing the poles of $\Gamma(b_j+s), j=1, 2, \ldots, m$ and $i=\sqrt{-1}$.

3. Brief Exposition of Various Approaches

Approach 1 : Method of Direct Integration

This method is the oldest one and is due to Wilks [28] Simple substitutions and direct integration give the desired results.

(*i*) *When* $p=2$:

Writing first the joint distribution of v_0 and v_1 with the help of (1.14) and substituting $v_0 v_1 = V_2$ and $v_0 = z$, we integrate it for z with the use (2.3) and then use (2.1). The distribution of V_2 in the non-central linear case is finally as follows:

$$\frac{e^{-\frac{k2}{2}} V_2^{\frac{n-3}{2}}}{\sqrt{2\pi}\Gamma(n-1)} \sum_{r=0}^{\infty} \left[\frac{\left(\frac{k^2}{4}\right)^r V_2^{\frac{1}{2}(r+\frac{1}{2})} \Gamma(n/2)}{r \,!\, \Gamma\left(\frac{n}{2}+r\right)} K_{r+\frac{1}{2}} (\sqrt{V_2}) \right] dV_2, \quad 0 \leqslant V_2 < \infty$$

Setting $k^2=0$ and using the result (42) due to Erdelyi [13] .the density of $\sqrt{V_2}$ in the central case is a chi-square density with $(2n-1)$ d.f.

Remark A : Thus it has been established that twice the square root of the product of independent central chi-squares with n and $(n-1)$ d.f. is again a chi-square variate with $2(n-1)$ d.f. This fact has been used quite frequently by researchers in obtaining the distribution of V_p for values of $p>2$.

(*ii*) *When* $p=3$, Using Remark A, we first write the joint distribution of v_0 and v_1 $v_2=u$ and then set u $v_0=v_3$ and $v_0=x$. Finally, integrating over x with the help of (2.2), the distribution of v_3 $(=v_0\, v_1\, v_2)$ is as follows :

$$\frac{e^{-k^2/2}\, V_3^{\frac{n-4}{2}}}{2^{n/2}\Gamma\left(\frac{n}{2}\right) \Gamma(n-2)} \sum_{r=0}^{\infty} \left[\frac{\left(\frac{k^2}{2}\right)^r \Gamma\left(\frac{n}{2}\right)}{r\,!\,\Gamma\left(\frac{n}{2}+r\right)} Lr\left(\frac{\sqrt{V_3}}{2}\right) \right] dV_3$$

$$0 \leqslant V_3 < \infty \qquad \ldots (3.2)$$

The distribution of V_3 in the central case is found by setting $k^2=0$ (Bagai [2].

Using the same procedure, the distribution of V_p for $p=4$, 5,..., can be found. For values of $p=4$ (1) 12, the reader is referred to Bagai [2] [3] [4] [5].

Approach 2 : Method of Inverse Mellin Transforms

The h^{th} order moment of the statistic, V_p, is given by

$$\mu'_h = \left(\prod_{i=1}^{p-1} \frac{2^h \Gamma\left(\frac{n-i}{2}+h\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right) 2^h e^{\frac{-k^2}{2}}$$

$$\sum_{r=0}^{\infty} \left[\frac{(k^2/2)^r \Gamma\left(\frac{n}{2}+r+h\right)}{r!\,\Gamma\left(\frac{n}{2}+r\right)} \right]$$

Substituting $h+(n-p+1)/2=t$ and using (2.4), the distribution of V_p, with little algebra, in the non-central linear case is as follows :

$$\frac{V_p^{(n-p-1)/2}\, e^{\frac{-k^2}{2}}}{2^{p(n-p+1)/2} \prod_{i=0}^{p-1} \Gamma\left(\frac{n-i}{2}\right)} \sum_{r=1}^{\infty} \left\{ \frac{(k^2/2)^r}{r!\,\Gamma\left(\frac{n}{2}+r\right)} \right.$$

$$\left. \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[V_p^{-t}\, \Gamma\left(t+r+\frac{p-1}{2}\right) \prod_{i=1}^{p-1} \Gamma\left(t+\frac{p-i-1}{2}\right)\right]\right\} dt \quad \text{...(3 3)}$$

$$0 \leqslant V_p < \infty$$

With the help of (3.3) we now illustrate Consul's method of finding the distribution of V_p for some values of p.

(*i*) *When* $p=2$

Setting $p=2$ in (3.3) and using (2.4*c*), the distribution of V_2 is given as follows :—

$$\frac{V_2^{(n-3)/2}\, e^{\frac{-k^2}{2}}}{2^{n-3/2}\,\Gamma\left(\frac{n-1}{2}\right)} \sum_{r=0}^{\infty} \left[\frac{(k^2/4)^r\, V_2^{(r+1/2)/2}}{r!\,\Gamma\left(\frac{n}{2}+r\right)} K_{r+\frac{1}{2}} \right.$$

$$\left. \sqrt{(V_2)} \right] dV_2\,; \quad 0 \leqslant V_2 < \infty \qquad \text{...(3.4)}$$

where $K_{r+\frac{1}{2}}(\cdot\ )$ is defined as in (2.3).

The central case is obtained by setting $k^2=0$.

(*ii*) *When* $p=3$

Setting $p=3$ in (3.3), using (2.4), (2.4*a*) and (2.4*c*) and simplifying with the help of (2.1), (2.2) and (2.2*a*), the distribution of V_3 is as follows :

$$\frac{V_3^{(n-4)/2} e^{\frac{-k^2}{2}}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)\Gamma(n-2)} \sum_{r=0}^{\infty} \left[\frac{(k^2/2)^r \Gamma\left(\frac{n}{2}\right)}{r!\,\Gamma\left(\frac{n}{2}+r\right)} L_r\ (\sqrt{V_3/2})\right] dV_3$$

$$0 \leqslant V_3 < \infty \qquad \ldots(3.5)$$

The same technique has been used by Consul [11] in obtaining the distribution of V_p for values of $p=4(1)\,7$.

Method of Inverse Mellin Transforms and Use of Meijer G-function :

Using (2.4), (2.5) and (3.3), the density of V_p in the non-central linear case is given by

$$f(V_p) = \frac{e^{\frac{-k^2}{2}}}{\prod_{j=1}^{p-1}\left(\frac{n-j}{2}\right)} \sum_{i=0}^{\infty} \left[\frac{(k^2/2)^r\, V_p^{-1}}{r!\,\Gamma\left(\frac{n}{2}+r\right)} G_{0,p}^{p,0}\right.$$

$$\left.\left(2^{-p}V_p \mid \frac{n}{2}+r, \frac{n-1}{2}, \ldots, \frac{n-p+1}{2}\right)\right];\ 0 \leqslant V_p < \infty \qquad \ldots(3.6)$$

Setting $k^2=0$, the density in the central case is given by

$$f(V_p) = \frac{1}{\pi \Gamma_{j=0}^{p-1}\left(\frac{n-j}{2}\right)} G_{0,p}^{p,0}\left(2^{-p}\, V_p \mid \frac{n}{2}, \frac{n-1}{2}, \ldots \frac{n-p+1}{2}\right)$$

$$0 \leqslant V_p < \infty \qquad \ldots(3.7)$$

The density function in central case has again been obtained by Mathai, Saxena and Rathie in a computable form for $p=2$, 3, 4 and that in the non-central case for $p=2$ with available results on Meijer G-function. For higher values of p, the $G_{0,p}^{p,0}$ – function has been shown by them in a computable form with the use of calculus of residues, the ψ – function and the Riemenn Zeta function. For some particular values of p their results agree with those of Bagai [2] [3] [4] and Consul [11].

Bagai [6] [7] [8] [9] also found the distribution of V_p in the non-central linear case in its most general form *i.e.* in terms of Meijer G—function $G^{p,0}_{0,p}\,(x/..)$. To simplify the work, he further gives the distribution in two alternative forms *i.e.* separately when p is even ($=2r$, say) and when p is odd ($=2r+1$, say). Further, reduction formulae also have beengiven by him from which the distribution of V_p for higher successive values of p can be conveniently derived. Bagai has actually worked out the derivation for the cases $p=2(1)\,12$ and his results for each p come out to be the same as already determined by Bagai [2], [5] and Consul [11] [12]. He claims that his reduction formulae can be utilized as a feed back in computation of numerical values of the central V_p for the successive higher values of p.

REFERENCES

[1] Anderson, T.W. (1946) : "The non-central Wishart distribution and certain problems of multivariate statistics." *Ann. Math. Stat.* **17**, 409-431.

[2] Bagai, O.P. (1962*a*) : "Distribution of the determinant of the sum of product matrix in the non-central linear case for some values of p". *Sankhya* **24**. 55-62.

[3] Bagai, O.P. (1962*b*) : "Statistics proposed for various tests of hypotheses and their distributions in particular cases". *Sankhyā* **24**, 409-418.

[4] Bagai. O.P. (1964) : "Limiting distribution of some statistics in multivariate analysis of variance." *Sankhyā*, **26**, 271-278.

[5] Bagai, O.P. (1965) : "The distribution of the generalized variance." *Ann. Math. Stat.* **36**, 120-130.

[6] Bagai, O.P. (1972*a*) : "On the exact distribution of some p-variate test statistics with the use of G-function." *Sankhyā*, **34**, 171-186.

[7] Bagai, O.P. (1972*b*) : "On the exact distribution of the (pxp) generalized sample variance in the non-central linear case for any p." *Statistics*. **32**, 65-74.

[8] Bagai, O.P. (1972*c*) : "On the exact distribution of the ratio of (pxp) generalized sample variance to any of its principal minors." *Estadistica*, **23** 45-56.

[9] Bagai, O.P. (1974) : "Distributions concerning generalized variances and their ratio". *Jour. Indian Stat. Assoc.* **12**, 7-19.

[10] Constantine, A.G. (1963) : "Some non-central distribution problems in multivariate analysis." *Ann. Math. State.* **34**, 1270-85.

[11] Consul P.C. (1964) : "On the limiting distributions of some statistics proposed for tests of hypotheses." *Sankhyā*, **26**, 279-286.

[12] Consul, P.C. (1965) : 'Distribution of the determinant of the sum of product matrix in the non-central linear case." *Mathematische Nachrichten*, **28**, 169-179.

[13] Erdélyi (1953) : *Higher Transcendental Functions* Vol. I, II McGraw-Hill, New York.

[14] Erdélyi (1954) : *Tables of Integral Transform* Vol. I,II, McGraw-Hill, New York.

[15] Herz, Corl. S. (1955) : "Bessel functions of matrix arguments." *Ann. of Math.* **61**. 474-523.

[16] James, A.T. (1954) : "Normal multivariate analysis and the orthogonal group". *Ann. Math. Stat.* **25**, 40-75.

[17] James, A.T. (1955) : "The non-central Wishart distribution." *Proc. Roy. Soc. London Series A* **229**, 264-366.

[18] James, A.T. (1961) : "The distribution of non-central means with known covariance. *Ann. Math. Stat* 32, 874-882.

[19] Mathai, A.M. (1972) : "The exact distribution of three multivariate Statistics with Wilks concept of generalized variance". *Sankhyā*, **34**, 161-170.

[20] Mathai A.M. (1973) : "A review of the different techniques used for deriving the exact distributions of multivariate test criteria." *Sankhyā*, **35**, 39-60.

[21] Mathai, A.M. and Rathie P.N. (1971) "The excat distribution of Wilks generalized variance in the non-central linear cases." *Sankhyā*, **33**, 45-60.

[22] Mathai, A.M. and Saxena R.K. (1968) "Exact distributions of multivariate test criteria." *Unpublished report.*

[23] Nair, U.S. (1938) : "The application of the moment function in the study of distribution laws in statistics." *Biometrika*, **30**, 274-294.

[24] Nanda, D.N. (1948) : "Limiting distributions of a root of a determinantal equation". *Ann. Math. Stat.* **19**, 340-350.

[25] Pillai, K.C.S. (1955) : "Some new criteria in multivariate analysis." *Ann. Math. Stat.* **26**, 117-121.

[26] Titchmarsh (1948) : *An Introduction to the Theory of Fourier Intergrals.*

[27] Whittaker, E.T. and Watson, O.N. (1927) : *A course of Modern Analysis.* 4th Ed., Cambridge University Press, Cambridge.

[28] Wilks, S.S. (1932) : "Certain generalizations in the analysis of variance." *Biometrika*, **26**, 471-494.